

Quantum and super-quantum group related to the Alexander–Conway polynomial

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We describe the quasitriangular structure (universal R -matrix) on the nonstandard quantum group $U_q(H_1, H_2, X^\pm)$ associated to the Alexander–Conway matrix solution of the Yang–Baxter equation. We show that this Hopf algebra is connected with the super-Hopf algebra $U_q\mathfrak{gl}(1|1)$ by a general process of superization.

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1. Introduction

Often the term quantum group is associated with the quantum algebras $U_q\mathfrak{g}$ of Drinfeld and Jimbo, but the theory of quasitriangular Hopf algebras (quantum groups) covers other non-standard examples. New quantum groups can be generated with the quantum double construction of Drinfeld [1] or the FRT approach [2], though it is not believed that all known quantum groups follow necessarily from one of these two methods. Since only the FRT construction will be of our concern here we refer to it briefly. To any standard matrix solution R of the Quantum Yang–Baxter Equation (QYBE), the FRT approach associates two bialgebras $A(R)$ of “function algebra” type and $U(R)$ of “enveloping algebra” type that under certain conditions can lead to a Hopf algebra and a quantum group. This was done in the pioneering work of ref. [2] providing in this way a matrix description of the standard $U_q\mathfrak{g}$. However, the construction is far more general as was shown in ref. [3]. In this reference it is established that for quite

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general solutions R it is possible to make $A(R)$ formally into a Hopf algebra and that the resulting quantum group of enveloping algebra type is necessarily quasi-triangular, i.e., possesses a “universal R -matrix”. This indicates that the FRT construction can be extended to a large class of new quantum groups since many “non-standard” solutions R are known.

Here we develop an example of this type. We study the Hopf algebra $U = U_q(H_1, H_2, X^\pm)$ associated to the non-standard solution

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix} \tag{1}$$

of the matrix QYBE $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$. This solution gives rise to the Alexander–Conway (AC) knot polynomial [4] and was first found in the context of statistical mechanics (q -state vertex model) in ref. [5]. Although the relevant new Hopf algebra structure $U_q(H_1, H_2, X^\pm)$ associated to this Alexander–Conway solution was computed in ref. [6] the questions of universal R -matrix and superization were left open. These two problems are solved here.

2. The quasitriangular Hopf algebra $U_q(H_1, H_2, X^\pm)$

The algebra $U_q(H_1, H_2, X^\pm)$ is introduced by the following

Definition 1. $U_q(H_1, H_2, X^\pm)$ is the Hopf algebra generated by the operators $\{H_1, H_2, X^\pm\}$, and relations

$$\begin{aligned} [H_1, H_2] &= 0, \\ [H_1, X^\pm] &= \pm 2X^\pm, \quad [H_2, X^\pm] = \mp 2X^\pm, \\ [X^+, X^-] &= \frac{K_1 K_2 - K_1^{-1} K_2^{-1}}{q - q^{-1}}, \quad (X^\pm)^2 = 0, \end{aligned} \tag{2}$$

where the operators K_1, K_2 are defined in terms of H_1, H_2 as

$$K_1 = q^{H_1/2}, \quad K_2 = e^{i(\pi/2)H_2} q^{H_2/2}, \tag{3}$$

and q is an arbitrary parameter. The comultiplication relations are given by

$$\begin{aligned} \Delta H_i &= H_i \otimes 1 + 1 \otimes H_i, \quad i = 1, 2, \\ \Delta X^+ &= X^+ \otimes K_1 + K_2^{-1} \otimes X^+, \\ \Delta X^- &= X^- \otimes K_2 + K_1^{-1} \otimes X^-, \end{aligned} \tag{4}$$

and the antipode S and counit ϵ are as follows:

$$S(H_i) = -H_i, \quad S(X^+) = -qK_1^{-1}K_2X^+, \quad S(X^-) = qK_1K_2^{-1}X^-,$$

$$\epsilon(H_i) = \epsilon(X^\pm) = 0.$$

The algebra mentioned above has been constructed using the FRT approach to obtain bialgebras, or Hopf algebras if possible, from matrix solutions of the QYBE [2]. More specifically, the set of relations (2) are $R_{21}L_1^\pm L_2^\pm = L_2^\pm L_1^\pm R_{21}$ and $R_{21}L_1^+ L_2^- = L_2^- L_1^+ R_{21}$, where L_1^\pm and L_2^\pm denote the operators $L_1^\pm = L^\pm \otimes 1$, $L_2^\pm = 1 \otimes L^\pm$ and L^+, L^- are the lower and upper triangular matrices with ansatz

$$L^+ = \begin{pmatrix} K_1 & 0 \\ (q - q^{-1})X^+ & K_2^{-1} \end{pmatrix}, \quad L^- = \begin{pmatrix} K_1^{-1} & -(q - q^{-1})X^- \\ 0 & K_2 \end{pmatrix}.$$

The matrix R_{21} is defined as PRP , with P is the permutation operator $P(a \otimes b) = b \otimes a$, which in this case has the form

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and R is the AC solution shown in (1). The coalgebra structure (4) comes from the relation $\Delta L_{ij}^\pm = \sum_k L_{ik}^\pm \otimes L_{kj}^\pm$, where L_{ij}^\pm are the matrix elements of L^\pm . The formulae $(X^\pm)^2 = 0$ exhibited in (2) are highly suggestive of a super-algebra with X^\pm odd and K_1, K_2 even operators. However, $U_q(H_1, H_2, X^\pm)$ is not a super-quantum group. The reason is simple. To extend the comultiplication Δ to products of generators we must use the bosonic manipulation $(a \otimes b)(c \otimes d) = ac \otimes bd$ with respect to which the map $\Delta: U \rightarrow U \otimes U$ is an algebra homomorphism. If U were a super-quantum group this relation should be replaced by

$$(a \otimes b)(c \otimes d) = (-1)^{\deg(b)\deg(c)} ac \otimes bd,$$

which is inconsistent with the relations in the algebra U . In other words, $U_q(H_1, H_2, X^\pm)$ reminds us of a super-quantum group but it is an ordinary bosonic one. We address this problem in the next section.

Let us present now the quasitriangular structure for this Hopf algebra $U_q(H_1, H_2, X^\pm)$. We recall that any Hopf algebra U is called *quasitriangular* if there is an invertible element \mathcal{R} in $U \otimes U$ that obeys the axioms [1]

$$\Delta'(a) = \mathcal{R} \Delta(a) \mathcal{R}^{-1} \tag{5}$$

for all a in U and

$$(\Delta \otimes \text{id}) \mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{23}, \quad (\text{id} \otimes \Delta) \mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{12}. \tag{6}$$

Here Δ' is defined as $\tau \circ \Delta$, where $\tau(x \otimes y) \mapsto y \otimes x$ and the object \mathcal{R} is known as the “universal R-matrix”. Equations (6) are evaluated in $U^{\otimes 3}$ and if we write \mathcal{R} as

the formal sum $\mathcal{R} = \sum_i a_i \otimes b_i$, the operators \mathcal{R}_{12} , \mathcal{R}_{13} and \mathcal{R}_{23} are defined by the expressions

$$\mathcal{R}_{12} = \sum_i a_i \otimes b_i \otimes 1, \quad \mathcal{R}_{13} = \sum_i a_i \otimes 1 \otimes b_i, \quad \mathcal{R}_{23} = \sum_i 1 \otimes a_i \otimes b_i.$$

The physical importance of \mathcal{R} is that it satisfies the quantum Yang–Baxter equation

$$\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12}$$

abstractly in the algebra U . Now we find such a quasitriangular structure.

Theorem 2. *The Hopf algebra $U_q(H_1, H_2, X^\pm)$ is quasitriangular with universal R -matrix*

$$\mathcal{R} = e^{-i(\pi/4)H_2 \otimes H_2} q^{1/4(H_1 \otimes H_1 - H_2 \otimes H_2)} (\mathbb{1} \otimes \mathbb{1} + (1 - q^2)E \otimes F), \tag{7}$$

where E and F denote the operators $K_2 X^+$ and $K_2^{-1} X^-$, respectively.

Proof. It is sufficient to show that this \mathcal{R} verifies Drinfeld’s axioms (5) and (6). □

This quasitriangular structure is crucial for numerous applications. In particular, the quantity $u = \sum_i S(b_i) a_i$ which implements the square of the antipode in the form $uau^{-1} = S^2(a)$ for all elements a of the quantum group is of special interest in the general theory. In this case u in $U_q(H_1, H_2, X^\pm)$ is given by

$$u = e^{i(\pi/4)H_2^2} q^{-1/4(H_1^2 - H_2^2)} (\mathbb{1} + (1 - q^2)K_1^{-1} K_2^{-1} FE).$$

Another motivation to evaluate the elements \mathcal{R} and u is that they are useful in providing link invariants.

We finish this section by referring to the *canonical representation* of $U_q(H_1, H_2, X^\pm)$ defined by

$$\begin{aligned} \rho(H_1) &= \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, & \rho(H_2) &= \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \\ \rho(X^+) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & \rho(X^-) &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{8}$$

In this canonical representation ρ we recover $\rho \otimes \rho(\mathcal{R}) = R$, the AC solution (1) of the introduction. This result is to be expected from the general theory of matrix quantum groups [3, sec. 3] and also serves as a check on our universal R -matrix (7).

3. Superization of $U_q(H_1, H_2, X^\pm)$

We have already pointed that the relations $(X^\pm)^2=0$ are surely indicative of some kind of super-Hopf algebra structure. Yet $U_q(H_1, H_2, X^\pm)$ is an ordinary Hopf algebra and therefore not a super one at all. We give now some insight into this puzzle by means of the transmutation theory of ref. [7]. This reference contains theorems which show how, under suitable circumstances, we can transform Hopf algebras into super-Hopf algebras and vice versa. The purpose of this section is to prove, using one of these theorems, that the superization of a quotient of $U_q(H_1, H_2, X^\pm)$ coincides with the super-quantum group $U_q\text{gl}(1|1)$. We begin by introducing the concept of superization of an ordinary Hopf algebra. If H is any Hopf algebra containing a group-like element g such that $g^2 = \mathbb{1}$, we can define its *superization* as the super-Hopf algebra \hat{H} with the following properties. As an algebra \hat{H} coincides with H and the counit also coincides. As far as the comultiplication, antipode and quasitriangular structure (if any) of H are concerned, all are modified to the super ones given by

$$\hat{\Delta}h = \sum_k x_k g^{-\text{deg}(y_k)} \otimes y_k, \quad \hat{S}(h) = g^{\text{deg}(h)} S(h),$$

$$\hat{\mathcal{R}} = \mathcal{R}_g^{-1} \sum_i a_i g^{-\text{deg}(b_i)} \otimes b_i.$$

Here h denotes an arbitrary element of H with comultiplication $\Delta h = \sum_k x_k \otimes y_k$, $\mathcal{R} = \sum_i a_i \otimes b_i$ is the universal R -matrix of H and the element \mathcal{R}_g is $\mathcal{R}_g = \mathcal{R}_g^{-1} = \mathbb{1} \otimes \mathbb{1} - 2p \otimes p$, with $p = (\mathbb{1} - g)/2$. When h is considered as an element of \hat{H} its grading $\text{deg}(h)$ is given by the action of g on h in the adjoint representation, that is to say, by $ghg^{-1} = \text{deg}(h) \cdot h$ on homogeneous elements.

In order to apply the above superization theorem to $U_q(H_1, H_2, X^\pm)$ we first note that in this algebra the role of g can be played by the operator $e^{i\pi H_2/2}$ introduced in (3). This element $g = e^{i\pi H_2/2}$ has the property that g^2 is central and group-like. Hence it is natural to impose the relation $g^2 = \mathbb{1}$ in the abstract algebra. Moreover, we know that $\rho(g^2) = \mathbb{1}$ in the representation (8), so this further relation is consistent with the canonical representation. Note that g itself commutes with K_1, K_2 and anticommutes with X^\pm . We have

Proposition 3. *Let $g = e^{i\pi H_2/2}$. The superization of the Hopf algebra $U_q(H_1, H_2, X^\pm)/g^2 = \mathbb{1}$ is the super-Hopf algebra $U_q\text{gl}(1|1)/e^{2\pi i N} = \mathbb{1}$. Here $U_q\text{gl}(1|1)$ is defined by generators C, N even and η, η^+ odd with relations*

$$[N, \eta] = -\eta, \quad [N, \eta^+] = \eta^+,$$

$$\{\eta, \eta^+\} = \frac{q^C - q^{-C}}{q - q^{-1}},$$

$$\eta^2 = 0, \quad (\eta^+)^2 = 0, \tag{9}$$

and the operator C central. The supercomultiplication is given by

$$\begin{aligned} \hat{\Delta}C &= C \otimes \mathbb{1} + \mathbb{1} \otimes C, & \hat{\Delta}N &= N \otimes \mathbb{1} + \mathbb{1} \otimes N, \\ \hat{\Delta}\eta &= \eta \otimes q^{C-N} + q^{-N} \otimes \eta, & \hat{\Delta}\eta^+ &= \eta^+ \otimes q^N + q^{-C+N} \otimes \eta^+. \end{aligned} \tag{10}$$

This $U_q\mathfrak{gl}(1|1)$ has a super-quasitriangular structure given by the expression

$$\hat{\mathcal{R}} = q^{-(C \otimes N + N \otimes C)} (\mathbb{1} \otimes \mathbb{1} + (1 - q^2) q^N \eta \otimes q^{-N} \eta^+). \tag{11}$$

Proof. The proof of this proposition follows from a straightforward application of the above superization construction applied to the quotient. According to this the emerging super-Hopf algebra is generated by H_1, H_2 even and X^\pm odd operators with relations (2) unchanged, supercomultiplication given by

$$\begin{aligned} \hat{\Delta}H_i &= H_i \otimes \mathbb{1} + \mathbb{1} \otimes H_i, \quad i = 1, 2, \\ \hat{\Delta}X^+ &= X^+ \otimes K_1 + K_2^{-1} g \otimes X^+, \\ \hat{\Delta}X^- &= X^- \otimes K_2 + K_1^{-1} g \otimes X^-, \end{aligned}$$

and superantipode and supercounit as follows:

$$\begin{aligned} \hat{S}(H_i) &= -H_i, & \hat{S}(X^+) &= -qK_1^{-1}K_2gX^+, & \hat{S}(X^-) &= qK_1K_2^{-1}gX^-, \\ \hat{\epsilon}(H_i) &= \hat{\epsilon}(X^\pm) = 0. \end{aligned}$$

In these expressions the K_i are defined as in (3). It is not difficult to recognize the resulting super-quantum group as $U_q\mathfrak{gl}(1|1)$ if we redefine the generators H_1, H_2, X^\pm as C, N, η, η^+ according to

$$C = (H_1 + H_2)/2, \quad N = H_2/2, \quad \eta = X^+, \quad \eta^+ = X^-g$$

(so that $q^C = K_1K_2g$ and $q^N = K_2g$). Note that a direct consequence of this definition is that C is central as stated. Introducing these definitions in the above relations we obtain (9), (10) and (11) without difficulty. Let us stress that the supercomultiplication (10) is an algebra homomorphism consistent with the relations (9), provided we use super manipulation. \square

The super-quantum group $U_q(H_1, H_2, X^\pm)$ has already been connected with the Alexander–Conway polynomial in a physical state in ref. [8], so the importance of this proposition lies in the fact that it solves precisely the suspected connection between $U_q(H_1, H_2, X^\pm)$ and $U_q\mathfrak{gl}(1|1)$.

We conclude this section by mentioning that the previous superization theorem is not the only existing method of turning quantum groups into super-quantum groups. This $U_q\mathfrak{gl}(1|1)$ can also be obtained by the *graded* FRT construction associated to the graded variant of the solution (1) [9]. This confirms that the superization of $U_q(H_1, H_2, X^\pm)$ shown here is fully compatible with existing ideas

of transforming solutions of the QYBE into solutions of the graded QYBE and finding their corresponding quantum groups.

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